On the Majority Game Chromatic Number of Trees

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Abstract

A majority coloring of a graph is a vertex coloring in which no vertex has more than half of its neighbors colored with its own color. The least number of colors required for such a coloring is the majority chromatic number. In the corresponding coloring game, two players alternately color vertices while maintaining the majority condition; the least number of colors for which the first player has a winning strategy is the majority game chromatic number $\mu_g(G)$ for the graph G. It is known that $\mu_g(G)$ is unbounded in general, while $\mu_g(T) \leq 3$ for complete binary trees. In this note we study $\mu_g(G)$ for trees, with emphasis on ternary trees. We introduce a configuration-based strategy for the first player, identify minimal local obstructions that could force a fourth color, and show that these configurations do not arise in ternary trees. As a consequence, $\mu_g(T) \leq 3$ for every ternary tree.

1 Introduction

Let $G = (V, E)$ be a graph. A coloring $c : V \to C$ is a *majority coloring* if for every vertex v,

$$|\{u \in N(v) : c(u) = c(v)\}| \leq \left\lfloor \frac{d(v)}{2} \right\rfloor .$$

The minimum size of such a color set is the *majority chromatic number* $\mu(G)$. Lovász [2] gave an elegant proof of the fact that $\mu(G) \leq 2$ for every finite graph G. This striking result also shifted attention naturally towards the setting of infinite graphs. The outstanding question in this area is the so-called Unfriendly Partition Conjecture [7], which states that every countably infinite graph admits a majority 2-coloring.

Beyond the basic vertex-coloring setting, several extensions have been studied. Majority colorings have been considered for directed graphs [6], where four colors suffice in general and three are conjectured to be enough. Edge versions of majority coloring have also been investigated [3]: for instance, every graph of minimum degree at least 2 admits a majority 4-edge-coloring, while stronger degree conditions guarantee 3 colors.

A game-theoretic variant, the *majority coloring game*, was introduced by Bosek, Grytczuk, and Jakóbczak [1]. Two players, Alice and Bob, alternately color vertices from a fixed palette, maintaining a valid partial majority coloring throughout. Alice wins if all vertices are eventually colored; otherwise Bob wins. The least number of colors guaranteeing a win for Alice is the *majority game chromatic number*, denoted $\mu_g(G)$.

In contrast to the static parameter $\mu(G)$, the game parameter $\mu_g(G)$ exhibits significantly richer behavior: it is unbounded even on bipartite graphs, yet satisfies general upper bounds such as $\mu_g(G) \leq \mathrm{col}_g(G)$, where $\mathrm{col}_g(G)$ is the *game coloring number* of G. For trees, it is known that $\mu_g(T) \leq 4$, and Bosek, Grytczuk, and Jakóbczak proved that $\mu_g(T) \leq 3$ for complete binary trees.

The present work continues this line of investigation for trees. In particular, we focus on ternary trees and establish the following:

Main result. *If T is a ternary tree, then $\mu_g(T) \leq 3$.*

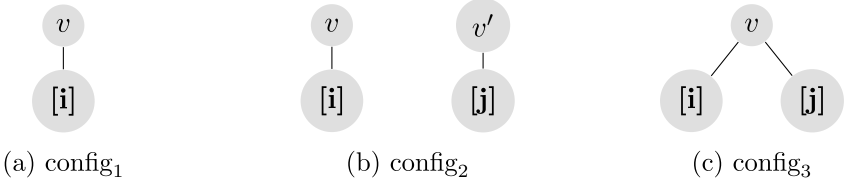
Our approach is configuration-based. We identify a small family of local obstruction patterns and show that Alice can play so as to prevent their occurrence. We then prove that any situation requiring a fourth color necessarily contains one of these forbidden configurations.

2 Strategy and local configurations

The key difficulty in the game is the emergence of *locally forced vertices*, i.e., uncolored vertices for which all available colors violate the majority condition. We control these by preventing the formation of certain local configurations.

We consider the following basic patterns. Here, by **[i]** we mean that the corresponding vertex, say v, is majority colored with $c(v) = i$.

- config_1: a majority-colored vertex with an uncolored parent,



(a) config_1 (b) config_2 (c) config_3

Figure 1: Basic configurations.

- config_2: two disjoint occurrences of config_1,
- config_3: two majority-colored siblings with distinct colors.

Alice's strategy is to immediately neutralize newly created majority vertices by coloring their parents whenever possible, and otherwise to assign colors so as to avoid repetition among siblings.

Alice's Strategy.

- If Bob's move creates a majority-colored vertex whose parent is uncolored, then Alice immediately colors that parent (if possible).
- If no such vertex is created, Alice colors a nearby vertex (typically a parent or sibling) so as to avoid repeating colors around an uncolored parent.
- When multiple color choices are available, she selects a color that is least frequent among the relevant neighboring vertices.

The strategy is designed to enforce the following invariant:

Alice does not create a majority-colored vertex whose parent remains uncolored.

This invariant turns out to be sufficient to eliminate all minimal obstruction configurations that could force the use of a fourth color. We next formalize the local configurations that capture the potential failure of this strategy.

Lemma 1. *Alice can play so as to avoid creating* config_1.

Proof sketch. If a move would create such a configuration, then a case analysis on the degree (at most 4 in a ternary tree) shows that at most one color is forbidden. Hence an alternative color is always available that avoids the configuration. □

Lemma 2. *If* config_1 *never occurs, then neither* config_2 *nor* config_3 *can occur.*

Proof sketch. Both configurations would require two majority vertices with uncolored parents. By the strategy, whenever such a vertex appears, its parent is colored immediately, preventing accumulation of multiple such vertices. $\square$

Corollary 1. *Under Alice's strategy, none of the configurations* config_1, config_2, *or* config_3 *appears.*

We next characterize when an uncolored vertex can become uncolorable using three colors.

Lemma 3. *If an uncolored vertex has no legal color among* $\{1, 2, 3\}$, *then either:*

- *it has three neighbors that are majority-colored with distinct colors, or*
- *one color is forbidden at the vertex itself while the remaining colors are forbidden by neighboring majority vertices.*

Proof sketch. A color becomes illegal either by violating the majority condition at the vertex or at a neighbor. The first can occur for at most one color, while the second requires a majority-colored neighbor of that color. $\square$

3 Main result

Theorem 1. *If* T *is a ternary tree, then* $\mu_g(T) \leq 3$.

Proof sketch. Suppose a fourth color is required at some vertex v. By the previous lemma, one of the critical obstruction patterns must occur in the neighborhood of v. In a ternary tree, these patterns necessarily involve either multiple majority-colored siblings or multiple vertices with uncolored parents. Both are excluded by the earlier configuration lemmas. Additionally, degree constraints prevent the accumulation of sufficiently many neighbors of the same color to force a violation. Hence no such obstruction can arise, and three colors suffice. $\square$

4 Concluding remarks

We have shown that $\mu_g(T) \leq 3$ for ternary trees via a configuration-based strategy. The approach extends almost verbatim to binary trees, thereby extending the result of Bosek, Grytczuk, and Jakóbczak for complete binary trees to all binary trees. We list several natural directions for further work below.

The number of basic configurations that need to be excluded increases for quaternary trees. Thus, the strategy outlined here does not immediately extend to r-ary trees for $r \geq 4$. We are presently unable to resolve the question of whether four colors are truly needed for Alice for such trees, and we raise this as an open question.

Question 1. *Does $\mu_g(T) \leq 3$ hold for r-ary trees T for $r \geq 4$?*

Although it is known that $\mu_g(T) \leq 4$ for any tree T, no explicit example is known for which the upper bound is attained. We believe that such examples exist, and can possibly be found among the class of r-ary trees for sufficiently large r. We ask below for explicit examples of such graphs.

Question 2. *Are there explicit constructions of trees T for which $\mu_g(T) = 4$?*

Lastly, we highlight that the behavior of the majority coloring parameter remains to be explored under variants of the coloring game, such as under restricted color reuse [4, 5] or alternative majority conditions [3].

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